

# SHARP ESTIMATE OF LOWER BOUND FOR THE FIRST EIGENVALUE IN THE LAPLACIAN OPERATOR ON COMPACT RIEMANNIAN MANIFOLDS

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**ABSTRACT.** The aim of this paper is give a simple proof of some results in [19] and [21], which are very deep studies in the sharp lower bound of the first eigenvalue in the Laplacian operator on compact Riemannian manifolds with nonnegative Ricci curvature. We also get a result about lower bound of the first Neumann eigenvalue in a special case. Indeed, our estimate of lower bound in the this case is optimal. Although the methods used in here due to [19] (or [21]) on the whole, to some extent we can tackle the singularity of test functions and also simplify greatly much calculation in these references. Maybe this provides another way to estimate eigenvalues.

## 1. INTRODUCTION

Suppose  $(M, g)$  is an  $n$ -dimensional compact Riemannian manifold with Ricci curvature satisfies

$$(1.1) \quad \text{Ric}(M) \geq (n-1)K$$

for some nonnegative constant  $K$ .

Unlike upper bound estimates, lower bound estimates for an eigenvalue is difficult to obtain. The study on the lower bound of the first positive eigenvalue in the Laplacian operator on compact Riemannian manifolds, can trace its history to a long time ago. In the meanwhile, there are many works in this area. Among these works, the results of Li ([4], [6]), Li-Yau ([5], [7]), Zhong-Yang [9], Yang [16], Ling [19]–[22], Ling-Lu [27], Shi-Zhang [30], Qian-Zhang-Zhu [31], Andrews-Ni [36], and Andrews-Clutterbuck [37], etc., are all very well known. It is difficult to describe all references in this field. So we just outline a portion of important works.

First of all, we state the following lower bound estimate of the first eigenvalue, which is due to Lichnerowicz 1958 [1] (also see Obata [2]) when  $M$  is a compact manifold without boundary. Under the same assumption (1.1), Escobar [12] proved that if a compact manifold has a weakly convex boundary, the first nonzero Neumann eigenvalue of  $M$  has the following lower bound (1.2) as well.

**Theorem 1.1.** (see Ling [16]) *Assume that  $\text{Ric}(M) \geq (n-1)K > 0$ . Let  $\lambda_1$  be the first positive eigenvalue on  $M$  (with either Dirichlet or Neumann boundary condition if  $\partial M \neq \emptyset$ ). If  $\partial M \neq \emptyset$ , we also assume that  $\partial M$  is of nonnegative mean curvature*

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$\text{tr}S \geq 0$  if  $\lambda_1$  is the first Dirichlet eigenvalue and  $\partial M$  is of nonnegative definite second fundamental form  $S \geq 0$  if  $\lambda_1$  is the first Neumann eigenvalue. Then

$$(1.2) \quad \lambda_1 \geq nK$$

This estimate provide no information when the above constant  $K$  vanishes. In such case, Li-Yau [5] and Zhong-Yang [12] provided another lower bound.

It is an interesting problem to find a unified lower bound of the first non-zero eigenvalue  $\lambda_1$  in terms of the lower bound  $(n-1)K$  of the Ricci curvature and the diameter  $d$ , the inscribed radius  $r$  and other geometric quantities, which do not vanish as  $K$  vanishes, of the manifold with positive Ricci curvature.

Later on, the maximum principle method which is rather different to that before, was first used by Li 1979 [4] in proving eigenvalue estimates for compact manifolds. From that time on, this method was then refined and used by many authors (e.g., Li-Yau [5], Zhong-Yang [9], Yang [16], etc.) for obtaining sharper eigenvalue estimates.

Soon after, using a improved maximum principle method, Li-Yau 1980 [5] derived the following beautiful result in the case when  $K = 0$ .

**Theorem 1.2. (Li – Yau).** *Let  $M$  be a compact Riemannian manifold,  $\partial M = \emptyset$ ,  $\text{Ric}(M) \geq 0$ , then  $\lambda_1 \geq \frac{\pi^2}{4d^2}$ , where  $d = \text{diam}(M)$  is the diameter of  $M$ .*

The above result was improved by Li 1982 [6] to  $\lambda_1 \geq \frac{\pi^2}{2d^2}$  in the case when  $K = 0$ . At one time Li had also conjectured that the first positive eigenvalue should satisfy

$$(1.3) \quad \lambda_1 \geq \frac{\pi^2}{d^2} + (n-1)K.$$

This conjecture greatly motivate many related studies in this area. It is necessary to mention those main results in the following.

Firstly, recall that the well-known Bonnet–Myers Theorem:

**Theorem 1.3. (Bonnet – Myers)** *Suppose that  $M$  is a  $n$ -dimensional complete Riemannian manifold with Ricci curvature bounded below by  $(n-1)K$  ( $K > 0$ ). Then  $M$  is compact, and its diameter  $d(M)$  satisfies the following estimate*

$$(1.4) \quad d(M) \leq \frac{\pi}{\sqrt{K}}.$$

Combining (1.3) with (1.4), we can deduce that (1.2) again. So (1.3) is usually regarded as the sharp lower bound on  $\lambda_1$  in terms of diameter for manifolds with Ricci curvature satisfies (1.1). Obviously, the optimal estimate to lower bound of the first eigenvalue is perfect and powerful. It seems that any further progress requires a refined gradient estimate which is relevant to the first eigenfunction.

By sharpening Li-Yau's method and giving a more delicate estimate, Zhong-Yang 1983 [9] improved this to the sharp estimate  $\lambda_1 \geq \frac{\pi^2}{d^2}$  in the case when  $K = 0$ . We now show Zhong-Yang's remarkable result as follows.

**Theorem 1.4. (Zhong – Yang).** *Let  $M$  be a compact Riemannian manifold without boundary, with nonnegative Ricci curvature and let  $d$  be the diameter of  $M$ . Then,*

$$(1.5) \quad \lambda_1 \geq \frac{\pi^2}{d^2}.$$

Next, we also remark here that the attempt to prove the so-called Li's conjecture would unify Yang-Zhong's estimates with Lichnerowicz's estimate. Several previous

efforts to prove (1.3) have been made, particularly towards improving inequalities of the form

$$(1.6) \quad \lambda_1 \geq \frac{\pi^2}{d^2} + \alpha(n-1)K$$

for some constant  $\alpha$ . These include works of Yang [16], Ling [19]–[22], Ling–Lu [27], Shi–Zhang [30], Qian–Zhang–Zhu [31], Ni [35], Andrews–Clutterbuck [34] and [37], also Andrews–Ni [36], etc. Now we proceed to state briefly some of these works in the following.

In a recent paper, following the similar methods in several previous works, but constructing a more complicated test function, Yang 1999 [16] has made a certain progress in Li’s conjecture, as shown by the following results.

**Theorem 1.5. (Yang)** *Let  $M^n$  be a closed Riemannian manifold with  $\text{Ric}(M^n) \geq (n-1)K \geq 0$  and diameter  $d$ . Then the first positive eigenvalue  $\lambda_1$  on  $M$  satisfies the lower bound*

$$\lambda_1 \geq \frac{\pi^2}{d^2} + \frac{(n-1)K}{4}.$$

**Theorem 1.6. (Yang)** *Let  $M^n$  be a compact manifold with nonempty boundary and with  $\text{Ric}(M^n) \geq (n-1)K \geq 0$ .*

(a) *Assume that the boundary  $\partial M$  is weakly convex, that is, the second fundamental form with respect to the outward normal is nonnegative. Then the first positive Neumann eigenvalue  $\lambda_1$  on  $M^n$  satisfies the same lower bound (1.2).*

(b) *Assume that the mean curvature with respect to the outward normal of the boundary  $\partial M$  is nonnegative. Then the first positive Dirichlet eigenvalue  $\lambda_1$  on  $M^n$  satisfies the lower bound estimate*

$$\lambda_1 \geq \frac{1}{4} \left[ \frac{\pi^2}{r^2} + (n-1)K \right].$$

where  $r$  is the inscribed radius for  $M^n$ .

In order to improve the known results above via the maximum principle method, one need to construct suitable test functions where detailed technical work is essential. In more recent two papers, Ling 2006 [19] and 2007 [21] give some new estimates on the lower bound and partially improve the lower bound above. The main results of those references are the following three theorems.

**Theorem 1.7. (Ling)** *If  $(M, g)$  is an  $n$ -dimensional compact Riemannian manifold with boundary. Suppose that Ricci curvature  $\text{Ric}(M)$  of  $M$  is bounded below by  $(n-1)K$  for some constant  $K > 0$*

$$\text{Ric}(M) \geq (n-1)K$$

*and that the mean curvature of the boundary  $\partial M$  with respect to the outward normal is nonnegative, then the first Dirichlet eigenvalue  $\lambda_1$  of the Laplacian  $\Delta$  of  $M$  has the following lower bound*

$$\lambda_1 \geq \frac{\pi^2}{\tilde{d}^2} + \frac{1}{2}(n-1)K,$$

where  $\tilde{d}$  is the diameter of the largest interior ball in  $M$ , that is,  $\tilde{d} = 2 \sup_{x \in M} \{\text{dist}(x, \partial M)\}$ .

**Theorem 1.8. (Ling)** *If  $M$  is an  $n$ -dimensional, compact Riemannian manifold that has an empty or nonempty boundary whose second fundamental form is nonnegative*

with respect to the outward normal (i.e., weakly convex). Suppose that Ricci curvature  $\text{Ric}(M)$  has a lower bound  $(n-1)K$  for some constant  $K > 0$ , that is

$$\text{Ric}(M) \geq (n-1)K > 0.$$

Then the first non-zero (closed or Neumann, which applies) eigenvalue  $\lambda_1$  of the Laplacian on  $M$  has the following lower bound

$$\lambda_1 \geq \frac{\pi^2}{d^2} + \frac{3}{8}(n-1)K \quad \text{for } n = 2$$

and

$$\lambda_1 \geq \frac{\pi^2}{d^2} + \frac{31}{100}(n-1)K \quad \text{for } n \geq 3,$$

where  $d$  is the diameter of  $M$ .

**Theorem 1.9. (Ling)** Under the conditions as in Theorem 1.8, if the manifold  $M$  has the symmetry that the minimum of the first eigenfunction is the negative of the maximum, i.e.,  $k = 1$  in (2.1), then the first nonzero (closed or Neumann, which applies) eigenvalue  $\lambda_1$  satisfies (1.6) with  $\alpha = \frac{1}{2}$ .

However, these are all entirely updated by the results of Shi–Zhang 2007 [30], Qian–Zhang–Zhu [31], as well as Andrews–Clutterbuck [37]. More precisely, Shi–Zhang [30] give the following result via using very differential method.

**Theorem 1.10. (Shi – Zhang)** Let  $M$  be a compact  $n$ -dimensional Riemannian manifold without boundary (or with convex boundary) and  $\text{Ric}(M) \geq (n-1)K$ . Then its first non-zero (Neumann) eigenvalue  $\lambda_1(M)$  satisfies

$$(1.7) \quad \lambda_1(M) \geq 4s(1-s)\frac{\pi^2}{d^2} + s(n-1)K \quad \text{for all } s \in (0, 1),$$

where  $d$  is the diameter of  $M$ .

Following the same argument in Shi–Zhang [30], Qian–Zhang–Zhu [31] generalize this result to the case when  $M$  is a Alexandrov space.

**Theorem 1.11. (Qian – Zhang – Zhu)** Let  $M$  be a compact  $n(\geq 2)$ -dimensional Alexandrov space without boundary and  $\text{Ric}(M) \geq (n-1)K$ . Then its first non-zero eigenvalue  $\lambda_1(M)$  satisfies (1.7), where  $d$  is the diameter of  $M$ .

Qian–Zhang–Zhu [31] also gives the following remarks.

*Remark 1.1.* (1) If let  $s = \frac{1}{2}$ , (1.11) becomes (1.6) with  $\alpha = \frac{1}{2}$ . This improves Chen–Wang’s result [14]–[15] in both  $K > 0$  and  $K < 0$ . It also improves Ling’s recent results in [21].

(2) If  $K > 0$ , Theorem 1.11 implies that

$$\lambda_1(M) \geq \frac{3}{4}\left[\frac{\pi^2}{d^2} + (n-1)K\right].$$

(3) If  $n \leq 5$  and  $K > 0$ , by choosing some suitable constant  $s$ , Qian–Zhang–Zhu [31] also get the following estimate

$$\lambda_1(M) \geq \frac{\pi^2}{d^2} + \frac{1}{2}(n-1)K + \frac{(n-1)^2 k^2 d^2}{16\pi^2}.$$

Recently Andrews–Clutterbuck [37] also get (1.6) with  $\alpha = \frac{1}{2}$ . Their contribution is the rather simple proof using the long-time behavior of the heat equation which is very likely much easier than the formerly available arguments. In particular, any trouble arising in previous works from possible asymmetry of the first eigenfunction is avoided in their argument. A similar argument proving the sharp lower bound for  $\lambda_1$  on a Bakry–Emery manifold had appeared in Andrews–Ni’s work [36]. Meanwhile, Andrews–Clutterbuck [37] also show that the inequality with  $\alpha = \frac{1}{2}$  is the best possible constant of this kind estimate, in other words, the Li conjecture is false. We refer the interested reader to consult [37] for some details.

Finally, notice that for manifolds with small diameter, Theorem 1.5–1.11 is better than the estimate (1.2). Therefore these results generalize Theorem 1.4. For more information in this direction, we refer to the excellent surveys by Ling–Lu [27], Qian–Zhang–Zhu [31], Ni [35], also Andrews–Clutterbuck [37], and so on, for further results of eigenvalue estimate and all the relevant references therein. To sum up, with the rapid development of geometric analysis, eigenvalue estimate on this stage is getting more and more important.

In present work we give a simple proof of Theorem 1.7 and 1.9, and also get the following result.

**Theorem 1.12.** *Under the assumptions as in Theorem 1.7, if the manifold  $M$  has the symmetry that the minimum of the first eigenfunction is the negative of the maximum, i.e.,  $k = 1$  in (2.1), then the first nonzero Dirichlet eigenvalue  $\lambda_1$  of the Laplacian on  $M$  satisfies (1.6) with  $\alpha = \frac{1}{2}$ .*

It is a key of this paper to constructing some suitable test function, even though we mainly use Zhong–Yang’s original approach [9] in our proof. Our argument is also based on several previous works, e.g., Li–Yau [5]–[6], Zhong–Yang [9], Ling [19] (or [21]), and so on. One interesting feature of our argument is that, it avoids various kinds of trouble from the singularity of  $|\nabla u|^2/(1 - u^2)$ , which is already present in those references. Although with many ways analogous to Ling [19] (or [21]), we can readily deal with the above singularity, and reduce the difficulty in calculation to some degree. Maybe this is another way to estimate eigenvalues.

The remaining part of this paper is organized as follows. In Section 2, we start by briefly introducing some terminologies and notations, that is consistent with [13], where the corresponding term are defined in more general setting. In Section 3, when  $\partial M \neq \emptyset$ , firstly we establish a technique lemma, which is recognized as another version of Lemma 2.2 in [16]. With aid of this lemma and the maximum principle, we then establish a rough estimate of  $F(\theta)$  (its definition in (2.6) below). A more precise estimate of  $F(\theta)$  is provided at the end of Section 4 via the method of barrier function. It turns out that this improved estimate is essential in the proof of Theorem 1.7, 1.9 and 1.12. Finally, as an application of the above estimate, the proof of all these theorems mentioned are presented in Section 5. In Section 6, we bring out an open problem in terms of the first eigenfunction. May be this problem associated with eigenvalue estimate as well.

## 2. NOTATIONS AND PRELIMINARIES

Let  $\{e_1, e_2, \dots, e_n\}$  be a local orthonormal frame field on  $M$ . We adopt the notation that subscripts in  $i, j$ , and  $k$ , with  $1 \leq i, j, k \leq n$ , mean covariant differentiations in the  $e_i, e_j$  and  $e_k$  directions respectively.

The Laplacian operator on  $M$  in term of local coordinates associated with the above orthonormal frame, is defined by differentiating once more in the direction of  $e_i$  and summing over  $i = 1, 2, \dots, n$ , i. e.,

$$\Delta u = \sum_i u_{ii}.$$

Denote by  $u$  the normalized eigenfunction with respect to the first eigenvalue  $-\lambda_1$  of  $\Delta$ . More precisely,

$$(2.1) \quad \begin{cases} \Delta u = -\lambda_1 u, \\ \max u = 1, \\ \min u = -k, \quad 0 < k \leq 1. \end{cases}$$

Throughout this paper, we always set

$$\theta(x) = \arcsin[u(x)], \quad \forall x \in M,$$

and define a subset of  $M$  as follows

$$\Sigma_* = \{x \in M : \theta(x) = \frac{\pi}{2} \quad \text{or} \quad \theta(x) = -\frac{\pi}{2} \quad \text{when} \quad k = 1\}.$$

Thus

$$u(x) = \sin[\theta(x)], \quad \forall x \in M$$

and

$$-\arcsin k \leq \theta(x) \leq \frac{\pi}{2}, \quad \forall x \in M.$$

Above terms shall apply unless otherwise mention.

By (2.1), a straight forward calculation shows that  $\theta(x)$  satisfies

$$(2.2) \quad \cos \theta \cdot \Delta \theta - \sin \theta \cdot |\nabla \theta|^2 = -\lambda_1 \sin \theta.$$

In particular,

$$(2.3) \quad \Delta \theta = \frac{\sin \theta}{\cos \theta} \cdot (|\nabla \theta|^2 - \lambda_1).$$

whenever  $x \in M \setminus \Sigma_*$ . From (2.2), we easily know that

$$(2.4) \quad |\nabla \theta|^2 = \lambda_1 \quad \text{as} \quad \theta = \frac{\pi}{2},$$

and when  $k = 1$ ,

$$(2.5) \quad |\nabla \theta|^2 = \lambda_1 \quad \text{as} \quad \theta = -\frac{\pi}{2}.$$

We also define a function  $F$  as follows

$$(2.6) \quad F(\theta_0) = \max_{x \in M, \theta(x) = \theta_0} |\nabla \theta(x)|^2$$

for all  $\theta_0 \in [-\arcsin k, \frac{\pi}{2})$  ( or  $(-\frac{\pi}{2}, \frac{\pi}{2})$  when  $k = 1$ ). Obviously,  $F$  is well-defined. Actually,  $F(\theta_0)$  is not something but an extreme value of  $f$  with condition  $\theta(x) = \theta_0$ . It is very easy to verify that  $F(\theta)$  is continuous in  $[-\arcsin k, \frac{\pi}{2})$  ( or  $(-\frac{\pi}{2}, \frac{\pi}{2})$  when  $k = 1$ ). Moreover, by (2.4) and (2.5), if we define

$$F(\frac{\pi}{2}) = F(\frac{\pi}{2} - 0) = \lambda_1,$$

and

$$F(-\frac{\pi}{2}) = F(-\frac{\pi}{2} + 0) = \lambda_1 \quad \text{when} \quad k = 1,$$

then  $F(\theta)$  can be extended a continuous function on  $[-\arcsin k, \frac{\pi}{2}]$  ( or  $[-\frac{\pi}{2}, \frac{\pi}{2}]$  when  $k = 1$ ).

### 3. A ROUGH ESTIMATE OF $|\nabla\theta|^2$

Firstly in a similar way owing to [8], [10], [13], also [16] and [19] (or [21]), we get the following lemma. Actually, it can be viewed as another version of Lemma 2.2 in [16].

**Lemma 3.1.** *Suppose that  $\partial M \neq \emptyset$ . Let  $G(\theta)$  be a function defined as follows*

$$G(x) = \frac{1}{2} |\nabla\theta(x)|^2 + g[\theta(x)], \quad \forall x \in M,$$

where  $g(\theta)$  is a smooth function defined on  $[-\arcsin k, \frac{\pi}{2}]$ . Then we have the following conclusions:

(1) *Assume that the mean curvature  $H$  of  $\partial M$  is nonnegative, also  $u$  satisfies the Dirichlet boundary condition, and  $g'(0) = 0$ . If  $G(x)$  arrives on its maximum at  $x_0 \in \partial M \setminus \Sigma_*$ , then  $\nabla G(x_0) = 0$ .*

(2) *Assume that the second fundamental form of  $\partial M$  is nonnegative with respect to the outward normal (i.e., weakly convex), also  $u$  satisfies the Neumann boundary condition. If  $G(x)$  attains its maximum at  $x_0 \in \partial M \setminus \Sigma_*$ , then  $\nabla\theta(x_0) = 0$ . Furthermore,  $\nabla G(x_0) = 0$ .*

*Proof.* Choose a local orthonormal frame  $\{e_1, e_2, \dots, e_n\}$  around  $x_0$  such that  $e_1$  is the unit normal  $\partial M$  pointing outward to  $M$ . We also denote below by  $\frac{\partial}{\partial x_1}$  the restriction on  $\partial M$  of the directional derivative corresponding to  $e_1$ .

The proof of (1): Clearly, the maximality of  $G(x_0)$  implies that

$$(3.1) \quad G_i(x_0) = 0 \quad \text{for} \quad 2 \leq i \leq n.$$

Since  $u$  satisfies the Dirichlet boundary condition, then

$$\theta|_{\partial M} = (\arcsin u)|_{\partial M} = 0.$$

Thus  $\theta_i(x_0) = 0$  for  $2 \leq i \leq n$ . We also derive from (2.3) that  $(\Delta\theta)|_{\partial M} = 0$ . Using these results in the following arguments, we have that at  $x_0$

$$\begin{aligned} \frac{1}{2} \frac{\partial(|\nabla\theta|^2)}{\partial x_1} &= \sum_{i=1}^n \theta_i \theta_{i1} = \theta_1 \theta_{11} = \theta_1 (\Delta\theta - \sum_{i=2}^n \theta_{ii}) \\ &= -\theta_1 \sum_{i=2}^n \theta_{ii} = -\theta_1 \sum_{i=2}^n (e_i e_i \theta - \nabla_{e_i} e_i \theta) \\ &= \theta_1 \sum_{i=2}^n \nabla_{e_i} e_i \theta = \theta_1 \sum_{i=2}^n \sum_{j=1}^n (\nabla_{e_i} e_i, e_j) \theta_j \\ &= \theta_1^2 \sum_{i=2}^n (\nabla_{e_i} e_i, e_1) = -\theta_1^2 \sum_{i=2}^n (\nabla_{e_i} e_1, e_i) \\ &= -\theta_1^2 \sum_{i=2}^n h_{ii} = -\theta_1^2 H \leq 0. \end{aligned}$$

Here  $h_{ij}$  and  $H = \sum_{i=2}^n h_{ii}$  are the second fundamental form and the mean curvature of  $\partial M$  relative to  $e_1$ , respectively.

$$\begin{aligned} \frac{\partial G}{\partial x_1}(x_0) &= \frac{1}{2} \frac{\partial(|\nabla \theta|^2)}{\partial x_1}(x_0) + g'[\theta(x_0)] \cdot \theta_1(x_0) \\ &\leq g'[\theta(x_0)] \cdot \theta_1(x_0) = g'(0) \cdot \theta_1(x_0) = 0. \end{aligned}$$

In addition, by the maximality of  $G(x)$  at  $x_0$ , we also have  $\frac{\partial G}{\partial x_1}(x_0) \geq 0$ . Thus

$$(3.2) \quad \frac{\partial G}{\partial x_1}(x_0) = 0.$$

Combining (3.1) with (3.2), we can get

$$\nabla G(x_0) = 0.$$

The proof of (2): By the maximality of  $G(x_0)$ , we also have

$$(3.3) \quad G_i(x_0) = 0 \quad \text{for } 2 \leq i \leq n$$

and

$$(3.4) \quad 0 \leq \frac{\partial G}{\partial x_1}(x_0) = \sum_{i=1}^n \theta_i(x_0) \cdot \theta_{i1}(x_0) + g'[\theta(x_0)] \cdot \theta_1(x_0)$$

In addition, since  $u$  satisfies the Neumann boundary condition, then

$$\theta_1 = \frac{1}{\sqrt{1-u^2}} \cdot u_1 = \frac{1}{\sqrt{1-u^2}} \cdot \frac{\partial u}{\partial x_1} = 0 \quad \text{on } \partial M.$$

Therefore,

$$(3.5) \quad \theta_1(x_0) = 0.$$

Putting (3.5) into (3.4), we then have

$$(3.6) \quad 0 \leq \frac{\partial G}{\partial x_1}(x_0) = \sum_{i=2}^n \theta_i(x_0) \cdot \theta_{i1}(x_0)$$

Notice that  $\theta_1(x_0) = 0$  and recall the definition of second fundamental form with respect to the outward normal, one can derive that, for  $2 \leq i \leq n$

$$\begin{aligned} \theta_{i1} &= e_i e_1 \theta - (\nabla_{e_i} e_1) \theta = e_i(\theta_1) - (\nabla_{e_i} e_1, e_j) \theta_j \\ &= -(\nabla_{e_i} e_1, e_j) \theta_j = - \sum_{j=2}^n h_{ij} \theta_j \quad \text{at } x_0, \end{aligned}$$

i.e., for  $2 \leq i \leq n$ ,

$$(3.7) \quad \theta_{i1} = - \sum_{j=2}^n h_{ij} \theta_j \quad \text{at } x_0,$$

where  $(h_{ij})_{2 \leq i, j \leq n}$  is the second fundamental form of  $\partial M$  relative to  $e_1$ . Putting (3.7) into (3.6), we can get

$$(3.8) \quad 0 \leq \frac{\partial G}{\partial x_1}(x_0) = - \sum_{i,j=2}^n \theta_i(x_0) h_{ij}(x_0) \theta_j(x_0) \leq 0,$$



since  $(h_{ij})_{2 \leq i, j \leq n}$  is nonnegative (i.e.,  $\partial M$  is weakly convex). Hence,  $\theta_i(x_0) = 0$  for  $2 \leq i \leq n$ . By (3.5) again, we have  $\nabla \theta(x_0) = 0$ . Finally,  $\nabla G(x_0) = 0$  follows from (3.3) and (3.8).

So far we finish the proof of this lemma.  $\square$

It was just as Zhong–Yang [9] had pointed out that the estimate of the upper bound of  $|\nabla \theta|^2$  plays an important role in the estimate of the lower bound for  $\lambda_1$ . In the following we establish a rough estimate for  $|\nabla \theta|^2$ .

**Lemma 3.2.** *Assume that  $\text{Ric}(M) \geq 0$ . The other assumption as in Theorem 1.1. In any case, the following estimate is valid.*

$$(3.9) \quad |\nabla \theta(x)|^2 \leq \lambda_1, \quad \forall x \in M.$$

Moreover,

$$(3.10) \quad F(\theta) \leq \lambda_1.$$

*Proof.* Suppose that  $|\nabla \theta|^2$  attains its local maximum at  $x_0$ . Clearly, (2.4) and (2.5) imply that (3.9) holds in the case:  $x_0 \in \Sigma_*$ . Without loss of generality, we may assume further that  $x_0 \in M \setminus \Sigma_*$  in the rest of the proof, thus  $\theta_0 = \theta(x_0) \in [-\arcsin k, \frac{\pi}{2})$  (or  $(-\frac{\pi}{2}, \frac{\pi}{2})$  when  $k = 1$ ). In the case of  $\partial M \neq \emptyset$ , with aid of Lemma 3.1, we conclude that if  $|\nabla \theta|^2$  arrive its maximum at  $x_0 \in M$ , then

$$(3.11) \quad \nabla(|\nabla \theta|^2) = 0 \quad \text{at } x_0.$$

no matter  $x_0 \in \partial M \setminus \Sigma_*$  or  $x_0 \in M \setminus (\partial M \cup \Sigma_*)$ . According to the maximum principle again, we easily show that

$$(3.12) \quad \Delta(|\nabla \theta|^2) \leq 0 \quad \text{at } x_0.$$

Applying the Bochner formula to  $\theta$ , we have

$$(3.13) \quad \frac{1}{2} \Delta(|\nabla \theta|^2) = |\nabla^2 \theta|^2 + \nabla \theta \cdot \nabla(\Delta \theta) + \text{Ric}(\nabla \theta, \nabla \theta),$$

where  $\text{Ric}(\nabla \theta, \nabla \theta)$  is the Ricci curvature along  $\nabla \theta$ . Substituting (2.3) into (3.13), we have

$$(3.14) \quad \begin{aligned} \frac{1}{2} \Delta(|\nabla \theta|^2) &= |\nabla^2 \theta|^2 + \nabla \theta \cdot \nabla \left[ \frac{\sin \theta}{\cos \theta} \cdot (|\nabla \theta|^2 - \lambda_1) \right] + \text{Ric}(\nabla \theta, \nabla \theta) \\ &= |\nabla^2 \theta|^2 + \nabla \theta \cdot \nabla \left( \frac{\sin \theta}{\cos \theta} \right) \cdot (|\nabla \theta|^2 - \lambda_1) \\ &\quad + \nabla \theta \cdot \frac{\sin \theta}{\cos \theta} \cdot \nabla(|\nabla \theta|^2) + \text{Ric}(\nabla \theta, \nabla \theta). \end{aligned}$$

A direct calculation leads to that

$$(3.15) \quad \nabla \left( \frac{\sin \theta}{\cos \theta} \right) = \frac{\nabla(\sin \theta) \cdot \cos \theta - \sin \theta \cdot \nabla(\cos \theta)}{\cos^2 \theta} = \frac{1}{\cos^2 \theta} \cdot \nabla \theta,$$

Putting (3.15) into (3.14), we obtain

$$(3.16) \quad \begin{aligned} \frac{1}{2} \Delta(|\nabla \theta|^2) &= |\nabla^2 \theta|^2 + \frac{1}{\cos^2 \theta} \cdot |\nabla \theta|^2 (|\nabla \theta|^2 - \lambda_1) \\ &\quad + \nabla \theta \cdot \frac{\sin \theta}{\cos \theta} \cdot \nabla(|\nabla \theta|^2) + \text{Ric}(\nabla \theta, \nabla \theta). \end{aligned}$$

By virtue of (3.11)–(3.12), we deduce from (3.16) that at  $x_0$

$$0 \geq |\nabla^2 \theta|^2 + \frac{1}{\cos^2 \theta} \cdot |\nabla \theta|^2 (|\nabla \theta|^2 - \lambda_1) + \text{Ric}(\nabla \theta, \nabla \theta).$$

Since  $\text{Ric}(M) \geq 0$ , the first term and the third term above can be taken away since they are nonnegative. Thus we obtain at  $x_0$

$$0 \geq \frac{1}{\cos^2 \theta} \cdot |\nabla \theta|^2 (|\nabla \theta|^2 - \lambda_1).$$

Dividing by  $|\nabla \theta|^2$  and multiplying by  $\cos^2 \theta$  successively, it follows that at  $x_0$

$$0 \geq |\nabla \theta|^2 - \lambda_1.$$

Hence we have

$$|\nabla \theta(x_0)|^2 \leq \lambda_1,$$

which implies the conclusion.  $\square$

#### 4. THE ESTIMATE OF $F(\theta)$

Now we are trying to get a more precise estimate on  $F(\theta)$  than Lemma 3.2. For this purpose, let us introduce the function  $Z(\theta) : [-\arcsin k, \frac{\pi}{2}] \mapsto \mathbb{R}$  such that

$$(4.1) \quad F(\theta) = \lambda_1 Z(\theta).$$

By Lemma 3.2, it is also easy to see that  $0 \leq Z(\theta) \leq 1$ . From now on we denote

$$\delta = \frac{(n-1)K}{2\lambda_1}.$$

It follows from (1.2) that

$$0 < \delta \leq \frac{n-1}{2n} < \frac{1}{2}.$$

**Lemma 4.1.** *Assume that  $\text{Ric}(M) \geq (n-1)K$  and the other conditions as in Theorem 1.1. If the function  $z : [-\arcsin k, \frac{\pi}{2}] \mapsto \mathbb{R}$  satisfies the following properties:*

- (1)  $z(\theta) \geq Z(\theta)$ ;
- (2) there exists some  $\theta_0 \in [-\arcsin k, \frac{\pi}{2}]$  (or  $(-\frac{\pi}{2}, \frac{\pi}{2})$  when  $k = 1$ ), such that  $z(\theta_0) = Z(\theta_0)$ ;
- (3)  $z'(0) = 0$ ;
- (4)  $z'(\theta_0) \sin \theta_0 > 0$ .

*Then the following estimate holds*

$$(4.2) \quad z(\theta_0) \leq 1 - \cos \theta_0 \sin \theta_0 \cdot z'(\theta_0) + \frac{\cos^2 \theta_0}{2} \cdot z''(\theta_0) - 2\delta \cos^2 \theta_0.$$

*Proof.* Set

$$f(x) = \frac{1}{2} \left\{ |\nabla \theta(x)|^2 - \lambda_1 z[\theta(x)] \right\}.$$

Obviously,  $f(x) \leq 0$  for all  $x \in M$ . By (2.6), we know that there exists some  $x_0 \in M \setminus \Sigma_*$  such that  $\theta(x_0) = \theta_0$  and  $F(\theta_0) = |\nabla \theta(x_0)|^2$ . Thus  $f$  achieves its maximum 0 at  $x_0$ , i. e.,

$$(4.3) \quad |\nabla \theta(x_0)|^2 = \lambda_1 Z(\theta_0) = \lambda_1 z(\theta_0).$$

By the same reason as in the proof of lemma 3.2, we always have

$$(4.4) \quad \nabla f(x_0) = 0,$$

no matter  $x_0 \in \partial M \setminus \Sigma_*$  or  $x_0 \in M \setminus (\partial M \cup \Sigma_*)$ . It is obvious that

$$(4.5) \quad \Delta f(x_0) \leq 0.$$

by the maximum principle again. Direct computation shows that

$$f_j = \sum_i \theta_i \cdot \theta_{ij} - \frac{\lambda_1}{2} z'(\theta) \cdot \theta_j,$$

that is

$$\nabla f = \frac{1}{2} [\nabla(|\nabla\theta|^2) - \lambda_1 z'(\theta) \cdot \nabla\theta] = \nabla\theta \cdot \nabla^2\theta - \frac{\lambda_1}{2} z'(\theta) \cdot \nabla\theta.$$

Since  $\nabla f = 0$  at  $x_0$ ,

$$(4.6) \quad \nabla(|\nabla\theta|^2) = 2\nabla\theta \cdot \nabla^2\theta = \lambda_1 z'(\theta_0) \cdot \nabla\theta \quad \text{at } x_0.$$

By directly calculating and applying (2.3), we also obtain

$$\begin{aligned} \frac{\lambda_1}{2} \Delta z &= \frac{\lambda_1}{2} \sum_j z_{jj} = \frac{\lambda_1}{2} \sum_j (z' \cdot \theta_j)_j \\ (4.7) \quad &= \frac{\lambda_1}{2} \sum_j (z'' \cdot \theta_j^2 + z' \cdot \theta_{jj}) = \frac{\lambda_1}{2} (z'' \cdot |\nabla\theta|^2 + z' \cdot \Delta\theta) \\ &= \frac{\lambda_1}{2} \left[ z'' \cdot |\nabla\theta|^2 + z' \cdot \frac{\sin\theta}{\cos\theta} \cdot (|\nabla\theta|^2 - \lambda_1) \right]. \end{aligned}$$

Combining (3.16) with (4.7), we hence obtain

$$\begin{aligned} \Delta f &= |\nabla^2\theta|^2 + \frac{1}{\cos^2\theta} \cdot |\nabla\theta|^2 (|\nabla\theta|^2 - \lambda_1) \\ &\quad + \nabla\theta \cdot \frac{\sin\theta}{\cos\theta} \cdot \nabla(|\nabla\theta|^2) + \text{Ric}(\nabla\theta, \nabla\theta) \\ &\quad - \frac{\lambda_1}{2} \left[ z'' \cdot |\nabla\theta|^2 + z' \cdot \frac{\sin\theta}{\cos\theta} \cdot (|\nabla\theta|^2 - \lambda_1) \right]. \end{aligned}$$

Recall that  $\text{Ric}(\nabla\theta, \nabla\theta) \geq (n-1)K |\nabla\theta|^2$ , we can get

$$\begin{aligned} \Delta f &= |\nabla^2\theta|^2 + \frac{1}{\cos^2\theta} \cdot |\nabla\theta|^2 (|\nabla\theta|^2 - \lambda_1) \\ (4.8) \quad &\quad + \nabla\theta \cdot \frac{\sin\theta}{\cos\theta} \cdot \nabla(|\nabla\theta|^2) + (n-1)K |\nabla\theta|^2 \\ &\quad - \frac{\lambda_1}{2} \left[ z'' \cdot |\nabla\theta|^2 + z' \cdot \frac{\sin\theta}{\cos\theta} \cdot (|\nabla\theta|^2 - \lambda_1) \right]. \end{aligned}$$

Substituting (4.6) into (4.8), it is easy to deduce that at  $x_0$

$$\begin{aligned} \Delta f &= |\nabla^2\theta|^2 + \frac{1}{\cos^2\theta} \cdot |\nabla\theta|^2 (|\nabla\theta|^2 - \lambda_1) \\ (4.9) \quad &\quad + \lambda_1 z' \cdot \frac{\sin\theta}{\cos\theta} \cdot |\nabla\theta|^2 + (n-1)K |\nabla\theta|^2 \\ &\quad - \frac{\lambda_1}{2} \left[ z'' \cdot |\nabla\theta|^2 + z' \cdot \frac{\sin\theta}{\cos\theta} \cdot (|\nabla\theta|^2 - \lambda_1) \right]. \end{aligned}$$

By virtue of (4.5), we derive from (4.9) that at  $x_0$

$$(4.10) \quad \begin{aligned} 0 \geq & |\nabla^2 \theta|^2 + \frac{1}{\cos^2 \theta} \cdot |\nabla \theta|^2 (|\nabla \theta|^2 - \lambda_1) \\ & + \lambda_1 z' \cdot \frac{\sin \theta}{\cos \theta} \cdot |\nabla \theta|^2 + (n-1)K |\nabla \theta|^2 \\ & - \frac{\lambda_1}{2} z'' \cdot |\nabla \theta|^2 + \frac{\lambda_1}{2} \cdot \frac{z' \sin \theta}{\cos \theta} \cdot (\lambda_1 - |\nabla \theta|^2). \end{aligned}$$

Obviously, condition (4) in this theorem and (3.9) imply that the last term in (4.10) is nonnegative. Thus the first term and the last term above can be discarded since they are nonnegative. We thus obtain that at  $x_0$

$$\begin{aligned} 0 \geq & \frac{1}{\cos^2 \theta} \cdot |\nabla \theta|^2 (|\nabla \theta|^2 - \lambda_1) + \lambda_1 z' \cdot \frac{\sin \theta}{\cos \theta} \cdot |\nabla \theta|^2 \\ & + (n-1)K |\nabla \theta|^2 - \frac{\lambda_1}{2} z'' \cdot |\nabla \theta|^2. \end{aligned}$$

After dividing by  $\lambda_1 |\nabla \theta|^2$ , multiplying by  $\cos^2 \theta$  and rearranging the terms successively, we are led to at  $x_0$

$$(4.11) \quad 0 \geq \frac{|\nabla \theta|^2}{\lambda_1} - 1 + z' \cdot \cos \theta \sin \theta - \frac{1}{2} z'' \cos^2 \theta + 2\delta \cos^2 \theta.$$

Therefore, using (4.3) we get at  $x_0$

$$(4.12) \quad 0 \geq z - 1 + z' \cdot \cos \theta \sin \theta - \frac{1}{2} z'' \cos^2 \theta + 2\delta \cos^2 \theta.$$

from which (4.2) follows easily. The proof is complete.  $\square$

We would like to point out that the remaining part of the present paper works exactly as in [9] (or [13]) and [19] (or [21]). For the completeness we briefly give a proof of Theorem 1.7, 1.9 and 1.12 below which only use the methods due to [9] and [19] (or [21]). We refer the interested reader to consult these references for more details.

**Lemma 4.2.** (see [19], or [21]) *Let*

$$(4.13) \quad \xi(\theta) = \frac{\cos^2 \theta + 2\theta \sin \theta \cos \theta + \theta^2 - \frac{\pi^2}{4}}{\cos^2 \theta} \quad \text{in} \quad \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

and  $\xi(\pm \frac{\pi}{2}) = 0$ . Then the function  $\xi$  satisfies the following

$$(4.14) \quad \frac{\cos^2 \theta}{2} \cdot \xi'' - \cos \theta \sin \theta \cdot \xi' - \xi = 2 \cos^2 \theta \quad \text{in} \quad \left(-\frac{\pi}{2}, \frac{\pi}{2}\right),$$

Moreover, the function  $\xi$  also has the following properties:

$$\xi(-\theta) = \xi(\theta), \quad \forall \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right);$$

$$\int_0^{\frac{\pi}{2}} \xi(\theta) d\theta = -\frac{\pi}{2};$$

$$\xi'(\theta) < 0 \quad \text{on} \quad \left(-\frac{\pi}{2}, 0\right) \quad \text{and} \quad \xi'(\theta) > 0 \quad \text{on} \quad \left(0, \frac{\pi}{2}\right);$$

$$(4.15) \quad 1 - \frac{\pi^2}{4} = \xi(0) \leq \xi(\theta) \leq \xi(\pm \frac{\pi}{2}) = 0 \quad \text{on} \quad \left[-\frac{\pi}{2}, \frac{\pi}{2}\right];$$

**Corollary 4.1.** *Let*

$$(4.16) \quad z(\theta) = 1 + \delta \xi(\theta).$$

*Then  $z$  satisfies the following*

$$(4.17) \quad \frac{\cos^2 \theta}{2} \cdot z''(\theta) - \cos \theta \sin \theta \cdot z'(\theta) - z(\theta) + 1 = 2\delta \cos^2 \theta \quad \text{in} \quad \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

$$z(\theta) > 0, \quad \forall \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right];$$

$$z'(0) = \delta \xi'(0);$$

$$z'(\theta) \sin \theta = \delta \xi'(\theta) \sin \theta \geq 0, \quad \forall \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

*Proof.* Using (4.15), we easily get

$$\begin{aligned} z(\theta) &= 1 + \delta \xi(\theta) \geq 1 + \delta \xi(0) \geq 1 + \delta \left(1 - \frac{\pi^2}{4}\right) \\ &> 1 + \frac{1}{2} \left(1 - \frac{\pi^2}{4}\right) = \frac{3}{2} - \frac{\pi^2}{8} \approx 0.26 > 0. \end{aligned}$$

In addition, by Lemma 4.2, it is direct to verify the other properties.  $\square$

Using Lemma 4.1 and Corollary 4.1 and the reduction to absurdity, we easily prove the following conclusion. For the reader's convenience, we give a proof below which is first due to [9], also due to [19] (or [21]).

**Lemma 4.3.** *Assume that  $Z(\theta)$  and  $z(\theta)$  are defined by (4.1) and (4.16), respectively. Then*

$$(4.18) \quad Z(\theta) \leq z(\theta).$$

*Proof.* Assume that (4.18) is not true. Since  $Z(\frac{\pi}{2}) = 1 = z(\frac{\pi}{2})$ , then there exists some  $\theta_0 \in [-\arcsin k, \frac{\pi}{2})$  (or  $(-\frac{\pi}{2}, \frac{\pi}{2})$  when  $k = 1$ ) such that

$$(4.19) \quad \sigma = Z(\theta_0) - z(\theta_0) = \max_{0 \leq \theta \leq \frac{\pi}{2}} \{Z(\theta) - z(\theta)\} > 0.$$

Set  $\tilde{z}(\theta) = z(\theta) + \sigma$ . Obviously,

$$\tilde{z}(\theta) = z(\theta) + \sigma \geq z(\theta) + [Z(\theta) - z(\theta)] = Z(\theta),$$

$$\tilde{z}(\theta_0) = z(\theta_0) + \sigma = Z(\theta_0),$$

$$\tilde{z}'(\theta) = z'(\theta),$$

$$\tilde{z}'(\theta_0) \sin \theta_0 = z'(\theta_0) \sin \theta_0 \geq 0.$$

In place of  $z(\theta)$  in Lemma 4.1 by  $\tilde{z}(\theta)$ , we deduce by Lemma 4.1 and (4.17) that

$$\begin{aligned} Z(\theta_0) &= \tilde{z}(\theta_0) \leq 1 - \cos \theta_0 \sin \theta_0 \cdot \tilde{z}'(\theta_0) + \frac{\cos^2 \theta_0}{2} \cdot \tilde{z}''(\theta_0) - 2\delta \cos^2 \theta_0 \\ &= 1 - \cos \theta_0 \sin \theta_0 \cdot z'(\theta_0) + \frac{\cos^2 \theta_0}{2} \cdot z''(\theta_0) - 2\delta \cos^2 \theta_0 = z(\theta_0). \end{aligned}$$

But this contradicts (4.19). The proof is complete.  $\square$

**Corollary 4.2.** *The assumption as in Theorem 1.1. In any case, the following estimate holds.*

$$(4.20) \quad F(\theta) \leq \lambda_1 z(\theta),$$

where  $F(\theta)$  and  $z(\theta)$  are defined by (2.6) and (4.16), respectively.

Our argument above establishes the inequality (4.20), which is an improved estimate of the upper bound for  $F(\theta)$  as required.

## 5. PROOF OF SOME THEOREM

Following the same argument in [19] (or [21]), we now use the estimate of  $F(\theta)$  to prove Theorem 1.7 as follows.

*Proof.* (4.20) implies that

$$\sqrt{\lambda_1} \geq \sqrt{\frac{|F(\theta)|}{z(\theta)}} = \sqrt{\frac{|F(\theta)|}{1 + \delta \xi(\theta)}} \geq \frac{|\nabla \theta|}{\sqrt{1 + \delta \xi(\theta)}},$$

i.e.,

$$(5.1) \quad \sqrt{\lambda_1} \geq \frac{|\nabla \theta|}{\sqrt{1 + \delta \xi(\theta)}},$$

where  $\xi(\theta)$  is defined by (4.13).

Take  $q_1 \in M$  such that  $\theta(q_1) = \frac{\pi}{2}$ . Choose  $q_2 \in \partial M$  such that  $\text{dist}(q_1, q_2) = \text{dist}(q_1, \partial M)$ . Clearly  $\theta(q_2) = 0$ . We denote by  $d'$  the length of a shortest curve  $\gamma$  which connects  $q_1$  with  $q_2$  on  $M$ . Let  $\tilde{d}$  be the diameter of the largest interior ball in  $M$  (i.e.,  $\tilde{d}/2$  is the inscribed radius of  $M$ ). Clearly,  $d' \leq \tilde{d}/2$ . Integrating both sides of (5.1) along the curve  $\gamma$ , we derive the following

$$\begin{aligned} \sqrt{\lambda_1} \frac{\tilde{d}}{2} &\geq \sqrt{\lambda_1} d' = \int_{\gamma} \sqrt{\lambda_1} ds \geq \int_{\gamma} \frac{1}{\sqrt{1 + \delta \xi(\theta)}} |\nabla \theta| ds \\ &\geq \int_{\gamma} \frac{1}{\sqrt{1 + \delta \xi(\theta)}} d\theta = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1 + \delta \xi(\theta)}} d\theta \\ &\geq \left( \int_0^{\frac{\pi}{2}} d\theta \right)^{\frac{3}{2}} / \left\{ \int_0^{\frac{\pi}{2}} [1 + \delta \xi(\theta)] d\theta \right\}^{\frac{1}{2}} \\ &= \left( \frac{\pi}{2} \right)^{\frac{3}{2}} / \left\{ \int_0^{\frac{\pi}{2}} [1 + \delta \xi(\theta)] d\theta \right\}^{\frac{1}{2}}. \end{aligned}$$

So, dividing by  $\frac{\tilde{d}}{2}$  and squaring the two sides successively, we have

$$\lambda_1 \geq \frac{\pi^3}{2\tilde{d}^2} / \int_0^{\frac{\pi}{2}} [1 + \delta \xi(\theta)] d\theta.$$

On the other hand,

$$\int_0^{\frac{\pi}{2}} [1 + \delta \xi(\theta)] d\theta = \frac{\pi}{2} + \delta \int_0^{\frac{\pi}{2}} \xi(\theta) d\theta = \frac{\pi}{2}(1 - \delta).$$

Hence, we conclude that

$$\lambda_1 \geq \frac{1}{1 - \delta} \cdot \frac{\pi^2}{\tilde{d}^2},$$

or, equivalently:

$$\lambda_1(1 - \delta) \geq \frac{\pi^2}{\tilde{d}^2}.$$

Therefore, we obtain

$$\lambda_1 \geq \frac{\pi^2}{d^2} + \lambda_1 \delta = \frac{\pi^2}{d^2} + \frac{(n-1)K}{2}.$$

This completes the proof.  $\square$

Using the similar argument as in the proof of Theorem 1.7, we prove Theorem 1.9 and Theorem 1.12 together in the following.

*Proof.* Take  $x_1, x_2 \in M$  such that  $\theta(x_1) = -\frac{\pi}{2}$ ,  $\theta(x_2) = \frac{\pi}{2}$ . We denote by  $d'$  the length of a shortest curve  $\gamma$  which connects  $x_1$  with  $x_2$  on  $M$ . Let  $d$  be the diameter of  $M$ . Clearly,  $d' \leq d$ . Integrating both sides of (5.1) along the curve  $\gamma$ , we derive the following

$$\begin{aligned} \sqrt{\lambda_1} d &\geq \sqrt{\lambda_1} d' = \int_{\gamma} \sqrt{\lambda_1} ds \geq \int_{\gamma} \frac{1}{\sqrt{1+z(\theta)}} |\nabla \theta| ds \\ &\geq \int_{\gamma} \frac{1}{\sqrt{1+z(\theta)}} d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{\sqrt{1+z(\theta)}} d\theta \\ &\geq \left( \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \right)^{\frac{3}{2}} / \left\{ \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} [1+z(\theta)] d\theta \right\}^{\frac{1}{2}} \\ &= \pi^{\frac{3}{2}} / \left\{ \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} [1+z(\theta)] d\theta \right\}^{\frac{1}{2}}. \end{aligned}$$

So we have

$$\lambda_1 \geq \frac{\pi^3}{d^2} / \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} [1+z(\theta)] d\theta.$$

In addition,

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} [1+z(\theta)] d\theta = \pi(1-\theta).$$

Thus

$$\lambda_1 \geq \frac{1}{1-\delta} \cdot \frac{\pi^2}{d^2},$$

or, equivalently:

$$\lambda_1(1-\delta) \geq \frac{\pi^2}{d^2}.$$

Hence, we get

$$\lambda_1 \geq \frac{\pi^2}{d^2} + \lambda_1 \delta = \frac{\pi^2}{d^2} + \frac{(n-1)K}{2}.$$

This is the required estimate. So far we complete the proof.  $\square$

## 6. FURTHER OPEN PROBLEM

At last we put forward a question

*Does every compact Riemannian manifold have the symmetry that the minimum of the first eigenfunction is the negative of the maximum, i.e.,  $k = 1$  in (2.1)?*

Indeed, this question was already implicit in [9]. Until now one do not know whether the first eigenfunction is symmetric or not. Perhaps it is more difficult to solve this question even if we add some assumption on the underlying Riemannian manifold. It is well know that if one can give an affirmative answer to this difficult question, or equivalently one can prove  $k = 1$ , then also one can obtain easily the optimal estimate

$$\lambda_1 \geq \frac{\pi^2}{d^2} + \frac{(n-1)K}{2}$$

by Theorem 1.9 and 1.12.

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